Lecture 5
Damping Ring Basics

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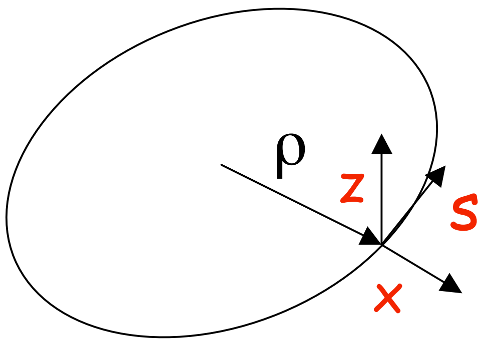
Outline

- Introduction
- Betatron motion
- Coupling
- Synchrotron motion
- Beam energy spread
- Beam emittance
- Radiation damping
- Intrabeam scattering

Betatron Motion

Reference trajectory

Mid-plane symmetry: magnetic field in the horizontal plane is perpendicular to the plane



$\rho(s)$ = local radius of curvature

$K(s)$ = local focusing strength

Particles are kept on a nearly circular trajectory by bending and focusing magnetic fields

The reference trajectory is the equilibrium closed orbit for a particle of momentum p_0. It is a sequence of straight lines and circular arcs (in bending magnets)

Quadrupoles act as focusing systems which produce small betatron oscillations around the reference trajectory

Multipolar expansion of Magnetic fields

For two dimensional fields the transverse field components B_x, B_y can be written as the Real and Imaginary part of an analytic function

$$\boxed{B_y + i\,B_x = f(z) = u(x,y) + i\,v(x,y) = \sum_{k=0}^{\infty} a_k\,(x + i\,y)^k = \sum_{k=0}^{\infty} r^k e^{ik\theta}}$$

Maxwell equations

$$divB = \frac{\partial B_y}{\partial y} + \frac{\partial B_x}{\partial x} = 0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$rot(B) = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

Cauchy-Riemann conditions

The analytic function can be expanded in multipolar series

Multipolar expansion of Magnetic fields

$$B = 0$$

drift space

$$B_y(x,y) + iB_x(x,y) = B_y$$

dipole term

$$+ \frac{\partial B_y}{\partial x}(x + iy)$$

quadrupole term

$$+ \frac{\partial^2 B_y}{\partial x^2}\left[(x^2 - y^2) + i(2xy)\right]$$

sextupole term

$$+ \frac{1}{6}\frac{\partial^3 B_y}{\partial x^3}\left[(x^3 - 3xy^2) + i(3x^2y - y^3)\right]$$

octupole term

Motion equation

The linearized betatron motion is governed by Hill's equation

$x'' + K_x(s)\, x = 0$ where $K_x = 1/\rho^2 - (\partial B_z/\partial x)\,/B\rho$

$z'' + K_z(s)\, z = 0$ and $K_z = (\partial B_z/\partial x)\,/B\rho$

The focusing functions are periodic:

$$K_{x,z}(s+L) = K_{x,z}(s)$$

Transfer matrices

Let $\mathbf{y}(s) = \begin{pmatrix} y(s) \\ y'(s) \end{pmatrix}$ be the "position vector"

$$\mathbf{y}(s) = M(s|s_0)\,\mathbf{y}(s_0)$$

where $M(s|s_0)$ is the betatron transfer matrix

The passage through a magnetic element can be described by a 2x2 matrix, which transforms the "position vector" of a particle before the element to the position vector after it

Solutions with constant K

$y'' + K(s)\, y = 0 \;\; ; \quad y = x \text{ or } z$

$Y(s) = a \cos(\sqrt{K}s + b)$ $\quad$ $K > 0$ focusing quad

$Y(s) = as + b$ $\quad$ $K = 0$ drift space

$Y(s) = a \cosh(\sqrt{-K}s + b)$ $\quad$ $K < 0$ defocusing quad

$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$K = 0$ drift space of length L

$$M_x = \begin{pmatrix} \cos\phi & \sin\phi/\sqrt{|K|} \\ -\sqrt{|K|}\,\sin\phi & \cos\phi \end{pmatrix}$$

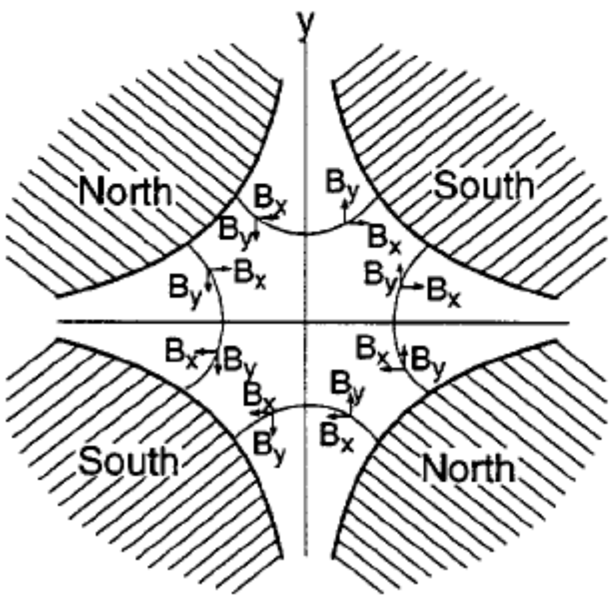
$K > 0$ focusing quad

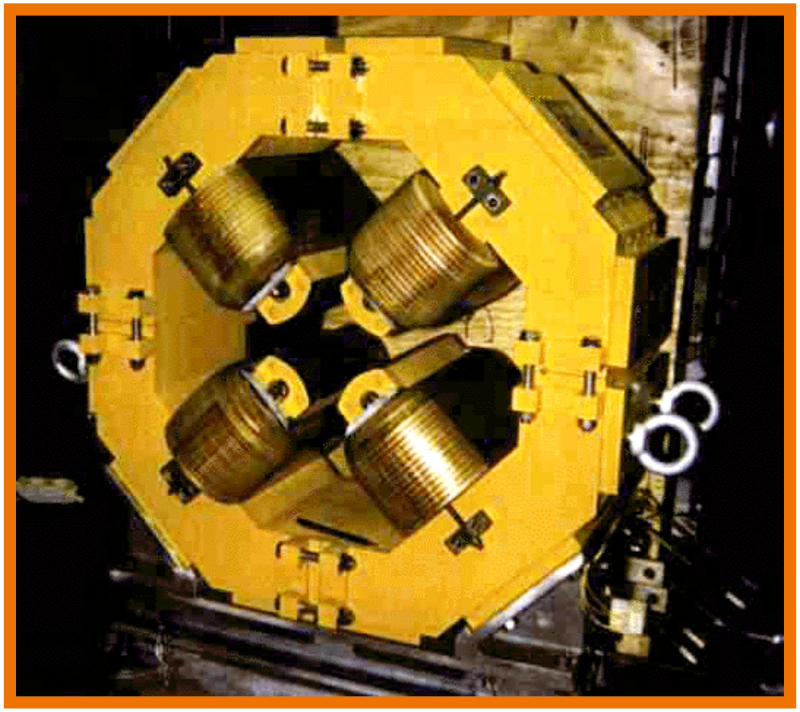
$\phi = s\sqrt{|K|}$

$$M_z = \begin{pmatrix} \cosh\phi & \sinh\phi/\sqrt{|K|} \\ -\sqrt{|K|}\,\sinh\phi & \cosh\phi \end{pmatrix}$$

$K < 0$ defocusing quad

Quadrupole





Strenght $K_x = (\partial B_z/\partial x)\,/B\rho$

$K_z = -(\partial B_z/\partial x)\,/B\rho$

Field

$B_x = (\partial B_z/\partial x)\cdot z$

$B_z = (\partial B_z/\partial x)\cdot x$

A quadrupole is always focusing in one plane and defocusing in the other one

One turn matrix

A ring is a sequence of N elements, and it can be represented by the product of the matrices of each element

$$M = M_N \cdot M_{N-1} \cdot \ldots M_3 \cdot M_2 \cdot M_1$$

The determinant of the matrices is: $\det(M_i)=1$

Periodicity condition: $M(s_1+L|s_1) = M(s_1)$

Map of m turns: $M(s1)^m$

One turn matrix

The one turn matrix can be written as:

Holds for any periodic cell

$$\boxed{M=\begin{pmatrix}\cos\phi+\alpha\sin\phi & \beta\sin\phi\\ -\gamma\sin\phi & \cos\phi-\alpha\sin\phi\end{pmatrix}=I\cos\phi+J\sin\phi}$$

with $$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};\quad J=\begin{pmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{pmatrix};\quad \mathrm{J}^2=-I;\quad \beta\gamma=1+\alpha^2$$

The map of n turns is: $M^n = I\cos(n\phi) + J\sin(n\phi)$

The condition for the stability of the betatron motion is:

$|\text{trace } M| = 2\cos\phi < 2$

α, β and γ are the Twiss parameters and ϕ is the betatron phase advance

One turn matrix

The transfer matrix from s_1 to s_2 can be written as:

$$M(s_2|s_1) = \begin{pmatrix} \sqrt{\beta_2/\beta_1}(\cos\psi + \alpha_1 \sin\psi) & \sqrt{\beta_1\beta_2}\sin\psi \\ \dfrac{-(1+\alpha_1\alpha_2)\sin\psi + (\alpha_1 - \alpha_2)\cos\psi}{\sqrt{\beta_1\beta_2}} & \sqrt{\beta_1/\beta_2}(\cos\psi - \alpha_2\sin\psi) \end{pmatrix}$$

$$\psi = \phi(s_1) - \phi(s_2)$$

And can be used to propagate the twiss functions:

$$\begin{aligned} \beta_2 &= M_{11}^2\beta_1 - 2M_{11}M_{12}\alpha_1 + M_{12}^2\gamma_1 \\ \alpha_2 &= -M_{11}M_{21}\beta_1 + (M_{11}M_{22} + M_{12}M_{21})\alpha_1 - M_{12}M_{22}\gamma_1 \\ \gamma_2 &= M_{21}^2\beta_1 - 2M_{21}M_{22}\alpha_1 + M_{22}^2\gamma_1 \end{aligned}$$

General solution of Hill's equation

$y'' + K(s)\,y = 0$; $y = x$ or z $\qquad K_{x,z}(s+L) = K_{x,z}(s)$

$$\boxed{y(s) = A\sqrt{\beta(s)}\cos(\varphi(s)+\delta)}$$

where A and δ are integration constants

This is the definition of the betatron oscillations and of the betatron function, which satisfies the following equation:

$$\frac{1}{2}\beta'' + K\beta - \frac{1}{\beta}\left[1+\left(\frac{\beta'}{2}\right)^2\right] = 0$$

Taking the first derivative:

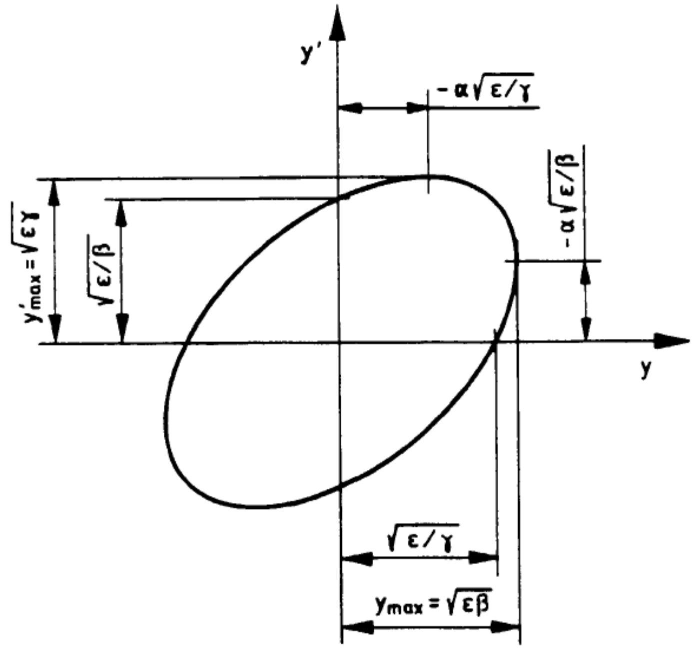
$$\boxed{y'(s) = \frac{A}{\sqrt{\beta(s)}}\left[\alpha(s)\cos(\varphi(s)+\delta) + \sin(\varphi(s)+\delta)\right]}$$

$$\boxed{\alpha(s) = -\beta'(s)/2 \;\; ; \;\; \gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)} \;\; ; \;\; \varphi(s) = \int \frac{1}{\beta(s)}ds}$$

Twiss Functions

$\phi(s)$ betatron phase advance

$\alpha(s)$, $\beta(s)$, $\gamma(s)$ Twiss functions



Since there are no friction terms in Hill's equation, the "energy" of the betatron oscillations is conserved during the motion

The trajectory of particle motion follows an ellipse described by the Courant-Snyder invariant

$$\boxed{\varepsilon = \frac{1}{\beta}\left[y^2 + (\alpha y + \beta y')^2\right] = \gamma y^2 + 2\alpha y y' + \beta y'^2}$$

The area of the ellipse is $\pi\varepsilon$, and it is constant along the ring

Twiss Functions

It is important to stress that the Twiss functions (also called optical functions) are periodic, their value depends only on the coordinate s along the ring, while the coordinates $y(s)$ and $y'(s)$ do not repeat after one revolution

$s = s + L$, L = ring circumference

If the storage ring is made of a number of identical cells, then the optical functions have the periodicity of the cells

The number of betatron oscillations in a full revolution is called the betatron tune:

$$\boxed{Q_y = \frac{\phi_y(L)}{2\pi} = \frac{1}{2\pi}\int_s^{s+L} \frac{ds}{\beta_y(s)}}$$

Dispersion function

A particle with a momentum deviation $\Delta p = p\text{-}p_0$ satisfies:

$$\boxed{x'' + K(s)x = \frac{1}{\rho}\frac{\Delta p}{p_0}}$$

The total deviation of the particle from the reference orbit is:

$$x(s) = x_\beta(s) + D(s)\Delta p/p_o$$

The particle with momentum deviation $\Delta p/p_o$ performs a betatron oscillation x_β with respect to an equilibrium orbit $D(s)\ \Delta p/p_o$

The dispersion function $D(s)$ satisfies:

$$\boxed{D'' + K(s)D = \frac{1}{\rho}}$$

With periodic boundary conditions

$$D(s+L) = D(s)\ \ ;\ \ D'(s+L) = D'(s)$$

Momentum Compaction

The total path length for an off-momentum particle differs from that for the on-momentum closed orbit by:

$$\boxed{\Delta C = \oint \frac{x}{\rho} ds = \left[\oint \frac{D(s)}{\rho} ds \right] \frac{\Delta p}{p}}$$

The momentum compaction factor a_c is defined by

$$\boxed{\alpha_c = \frac{\Delta C / C}{\Delta p / p} = \frac{1}{C} \oint \frac{D(s)}{\rho} ds = \frac{I_1}{C}}$$

The α_c parameter governs the longitudinal motion in storage rings. I_1 is the first of the radiation integrals that will be introduced in the following to describe the effects of synchrotron radiation emission

$$\boxed{I_1 = \oint \frac{D(s)}{\rho} ds}$$

Field errors and closed orbit

A small dipole field error produces a distortion of the reference orbit. The perturbation is approximated by an angular kick δ localized at the dipole center:

$$\boxed{\delta = \frac{1}{B\rho}\int B(s)ds}$$

Particles now perform betatron oscillations around a new orbit, which must be closed with the ring periodicity. In order to find the position vector at the perturbation we solve the system

$$\begin{vmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{vmatrix} \cdot \begin{vmatrix} y_c \\ y'_c \end{vmatrix} = \begin{vmatrix} y_c \\ y'_c - \delta \end{vmatrix} \quad ; \quad \Phi = 2\pi \mathrm{Q_y}$$

To get:

$$\boxed{\begin{aligned} y_c &= \frac{\beta\delta}{2\sin\pi Q_y}\cos\pi Q_y \\ y'_c &= \frac{\delta}{2\sin\pi Q_y}\left(\sin\pi Q_y - \alpha\cos\pi Q_y\right) \end{aligned}}$$

Closed orbit

Field errors and closed orbit

The c.o. can be propagated along the ring:

$$\boxed{y(s) = m_{11} y_c + m_{12} y'_c}$$

$$\boxed{y(s) = \frac{1}{2\left|\sin \pi Q_y\right|} \delta_k \sqrt{\beta(s)\beta(s_0)} \cos(\pi Q_y - \left|\phi(s) - \phi(s_0)\right|)}$$

The maxima occur at $\phi_\kappa = \phi(s_k) \text{-} \phi(s_0) = \pi Q_y + k\pi$ and are given by:

$$\boxed{y_k = \frac{1}{2\left|\sin \pi Q_y\right|} \delta \sqrt{\beta_k \beta(s_0)}}$$

When Q is integer the $\sin(\pi Q_y)$ term in y diverges and the motion becomes unstable

First order resonance: Q = n ; n=integer

Field errors and closed orbit

Dipole perturbations arise from:

$\delta = \Delta BL/(B\rho)$ field errors in bending magnets

$K_y L \Delta y_q$ quadrupole alignement errors Δy_q

Due to the linear nature of the Hill's equation the closed orbit of a ring is the sum of the orbits due to a large number of small errors

The expectation value of the amplitude due to a random distribution of dipole errors is:

$$\boxed{\left\langle y^2 \right\rangle = \frac{\beta(s)}{8\sin^2 \pi Q_y} \sum_i \beta_i \delta_i^{\,2} = \frac{\beta(s)\left\langle \beta \right\rangle}{8\sin^2 \pi Q_y} N \left\langle \delta^2 \right\rangle}$$

The amplitude of the closed orbit can be minimized by means of a set of dipole correctors properly distributed along the ring

Gradient errors

Let us consider a small perturbation of the focusing strength K:

$K=K_0+\Delta K$ with $K(s+L)=K(s)$

The effect of a small focusing perturbation can be represented by a "thin lens" matrix:

$$\begin{vmatrix} 1 & 0 \\ -\Delta K l & 1 \end{vmatrix}$$

The new one-turn transfer matrix is the product of the original matrix and the thin lens

$$\begin{vmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ -\Delta K l & 1 \end{vmatrix}$$

Gradient errors

Taking the trace of the matrix: $$\boxed{\cos(\Phi + \Delta\Phi) = \cos\Phi - \frac{1}{2}\beta\Delta K l \sin\Phi}$$

Since: $$\cos(\Phi + \Delta\Phi) = \cos\Delta\Phi\cos\Phi - \sin\Delta\Phi\sin\Phi$$

To first order:

$$\boxed{\Delta\Phi = \frac{1}{2}\beta\Delta K l \quad \text{and} \quad \Delta Q = \frac{1}{4\pi}\beta\Delta K l}$$

The result of gradient errors is a betatron tune shift, positive for a focusing quadrupole and negative for a defocusing one

For a distributed gradient error the tune shift is:

$$\boxed{\Delta Q = \frac{1}{4\pi}\oint \beta\Delta K ds}$$

Chromaticity

The focusing functions in Hill's equation depend on the momentum deviation via the radius of curvature ρ

$x'' + K_x(s)\, x = 0$ where $K_x = 1/\rho^2 - (\partial B_z/\partial x)/B\rho$

$z'' + K_z(s)\, z = 0$ and $K_z = (\partial B_z/\partial x)/B\rho$

$$K(\Delta p/p) \approx K(0)(1-\Delta p/p)$$

$$\Delta K_x \approx -K_x\, \Delta p/p$$

$$\Delta K_z \approx -K_z\, \Delta p/p$$

The dependence of the focusing strength on the momentum deviation Dp/p is called chromatic aberration

Using the formula for the gradient error we can calculate the betatron tune shift due to the chromatic aberration

$$\Delta Q_x = \frac{1}{4\pi}\oint \beta_x \Delta K ds \approx \left(-\frac{1}{4\pi}\oint \beta_x K_x ds\right)\frac{\Delta p}{p}$$

$$\Delta Q_z = \frac{1}{4\pi}\oint \beta_z \Delta K ds \approx \left(-\frac{1}{4\pi}\oint \beta_z K_z ds\right)\frac{\Delta p}{p}$$

Chromaticity

The chromaticity of a ring is defined as the derivative of the tunes with respect to momentum deviation

$$\boxed{C_x = \frac{\partial Q_x}{\partial \Delta p/p} \quad ; \quad C_z = \frac{\partial Q_z}{\partial \Delta p/p}}$$

Since the focusing strength is weaker for higher energy particles the betatron tune decreases and the chromaticity due to quadrupoles is always negative

Sextupole Chromaticity

Negative chromaticity can be the source of harmful instabilities: most machines have their chromaticity corrected to zero, or to slightly positive values, by means of sextupole magnets.

The field in a sextupole is:

$$\boxed{B_z = \frac{1}{2} B_2 (x^2 - z^2) \; ; \; B_x = B_2 xz}$$

The sextupole behaves as a quadrupole with a gradient proportional to the radial particle displacement

An off-momentum particle is displaced from the reference trajectory by $D(s)\Delta p/p$ and therefore experiences a gradient:

$$\boxed{\Delta K_x = \frac{1}{B\rho}\frac{\partial B_z}{\partial x} = \frac{B_2}{B\rho} x \quad ; \quad \Delta K_z = \frac{1}{B\rho}\frac{\partial B_x}{\partial z} = \frac{B_2}{B\rho} z}$$

Therefore the change in tune due to the sextupole is:

$$\boxed{\Delta Q = \frac{\Delta p / p}{4\pi} \oint \frac{B_2}{B\rho} D(s) \beta(s) ds}$$

Coupling

In the 4-dimensional phase space the matrix of an ideal horizontally focusing quadrupole is block diagonal and can be written as:

$$M = \begin{vmatrix} A_F & 0 \\ 0 & A_D \end{vmatrix}$$

Where A_F and A_D are 2x2 matrices

$$A_F = \begin{vmatrix} \cos(kl) & \sin(kl)/k \\ -k\sin(kl) & \cos(kl) \end{vmatrix} \;;\; A_D = \begin{vmatrix} \cosh(kl) & \sinh(kl)/k \\ k\sinh(kl) & \cosh(kl) \end{vmatrix}$$

If the quadrupole is rotated by an angle θ with respect to the horizontal symmetry plane, its matrix becomes:

$$M_R = \begin{vmatrix} c_1 A_F + c_2 A_D & c_3(A_F - A_D) \\ c_3(A_F - A_D) & c_2 A_F + c_1 A_D \end{vmatrix}$$

$$c_1 = \cos^2\theta$$
$$c_2 = \sin^2\theta$$
$$c_3 = -\sin\theta\cos\theta$$

Coupling

The rotated quadrupole matrix can be expressed in a more physical way as the product of a rotation of an angle θ by M and by a second rotation of $-\theta$:

$$M_R = R(\theta)\, M\, R(-\theta\,)$$

For $\theta = 45°$ the quadrupole is called skew quadrupole and its matrix is

$$\boxed{M_{skew} = \frac{1}{2}\begin{vmatrix} A_F + A_D & -A_F + A_D \\ -A_F + A_D & A_F + A_D \end{vmatrix}}$$

This matrix is not block diagonal as that of the ideal quadrupole. The motions of the particle in the two planes are no more independent. A kick in one plane excites betatron oscillations also in the other one and the motion is called "coupled". In a coupled machine, the oscillation amplitude is continuously transferred from the horizontal to the vertical plane and the other way around

Coupling

To describe the motion in a coupled machine we simplify the equation of motion assuming sinusoidal approximation

This can be done since the exchange of energy between the 2 planes is slow with respect to the betatron oscillation

$$\boxed{x'' + \kappa_x^2 x = -kz \;\; ; \;\; z'' + \kappa_z^2 z = -kx}$$

Where: $\kappa_x = Q_x/R$ and $\kappa_z = Q_z/R$ are the betatron phase advances assumed constant per unit length; R is the average ring radius

The coupling strength of the skew quadrupole, averaged over the ring, is then:

$$\boxed{k = \frac{1}{2\pi R}\frac{1}{B\rho}\oint\left(\frac{\partial B_x}{\partial x}\right)ds = \frac{-1}{2\pi R}\frac{1}{B\rho}\oint\left(\frac{\partial B_z}{\partial z}\right)ds}$$

Coupling

Let's try solutions of the form of uncoupled oscillations with an envelope term

$$\boxed{x = X(s)e^{i\kappa_x s}\ ;\ \ z = Z(s)e^{i\kappa_z s}}$$

For weak coupling conditions X and Z will be slowly varying functions and we neglect X" and Z"

We are interested to the strongest coupling resonance that occurs for $Q_x \approx Q_z$ so we put:

$\kappa \cong \kappa_x \cong \kappa_z$; and $\delta = \kappa_x - \kappa_z = (Q_x - Q_z)/R$

We get:

$$\boxed{X' = \frac{ie^{-i\delta s}}{2\kappa} kZ\ ;\ \ Z' = \frac{ie^{-i\delta s}}{2\kappa} kX}$$

Coupling

Solving the equations the general solution can be written:

$$x = e^{i\kappa_x s}\left[A e^{(i/2)(\eta-\delta)s} - B\frac{(\eta-\delta)}{k} e^{-(i/2)(\eta+\delta)s} \right] \quad ; \quad \eta = \sqrt{\delta^2 + (k/\kappa)^2}$$

$$z = e^{i\kappa_z s}\left[B e^{-(i/2)(\eta-\delta)s} - A\frac{(\eta-\delta)\kappa}{k} e^{(i/2)(\eta+\delta)s} \right]$$

A physical insight is obtained by taking the squares of the moduli of amplitude terms which form the envelopes of the oscillations:

$$XX^* = |X|^2 = |A|^2 + \frac{|B|^2(\eta-\delta)^2}{(k/\kappa)^2} + \frac{2|AB^*|(\eta-\delta)}{(k/\kappa)}\cos(\eta s)$$

$$ZZ^* = |Z|^2 = |B|^2 + \frac{|A|^2(\eta-\delta)^2}{(k/\kappa)^2} - \frac{2|AB^*|(\eta-\delta)}{(k/\kappa)}\cos(\eta s)$$

Coupling

There is a sinusoidal exchange of energy between the x and z planes with the interchange period:

$$\boxed{T = \frac{2\pi}{c\eta} = \frac{1}{f_{rev}} \frac{1}{R\sqrt{\delta^2 + (k/\kappa)^2}} = \frac{1}{f_{rev}} \frac{1}{\sqrt{\Delta^2 + |C|^2}}}$$

$$\Delta = Q_x - Q_z \quad ; \quad C = \frac{R^2}{Q} k = \frac{R}{2\pi Q} \frac{1}{B\rho} \oint \left(\frac{\partial B_x}{\partial x} \right) ds$$

And the total oscillation energy, given by the sum of the squares of the amplitudes, is conserved

$$\boxed{E_{\max} = |X|^2 + |Z|^2 = \frac{2\eta}{(\eta + \delta)} \left(|A|^2 + |B|^2 \right)}$$

Kick in the horizontal plane

When the beam is kicked in the H plane a vertical oscillation will start growing under the condition that $E_{max} = |X|^2 + |Z|^2$

The coherent oscillations and the exchange of amplitude will be visible to a pick-up in both planes

At s=0 it is:

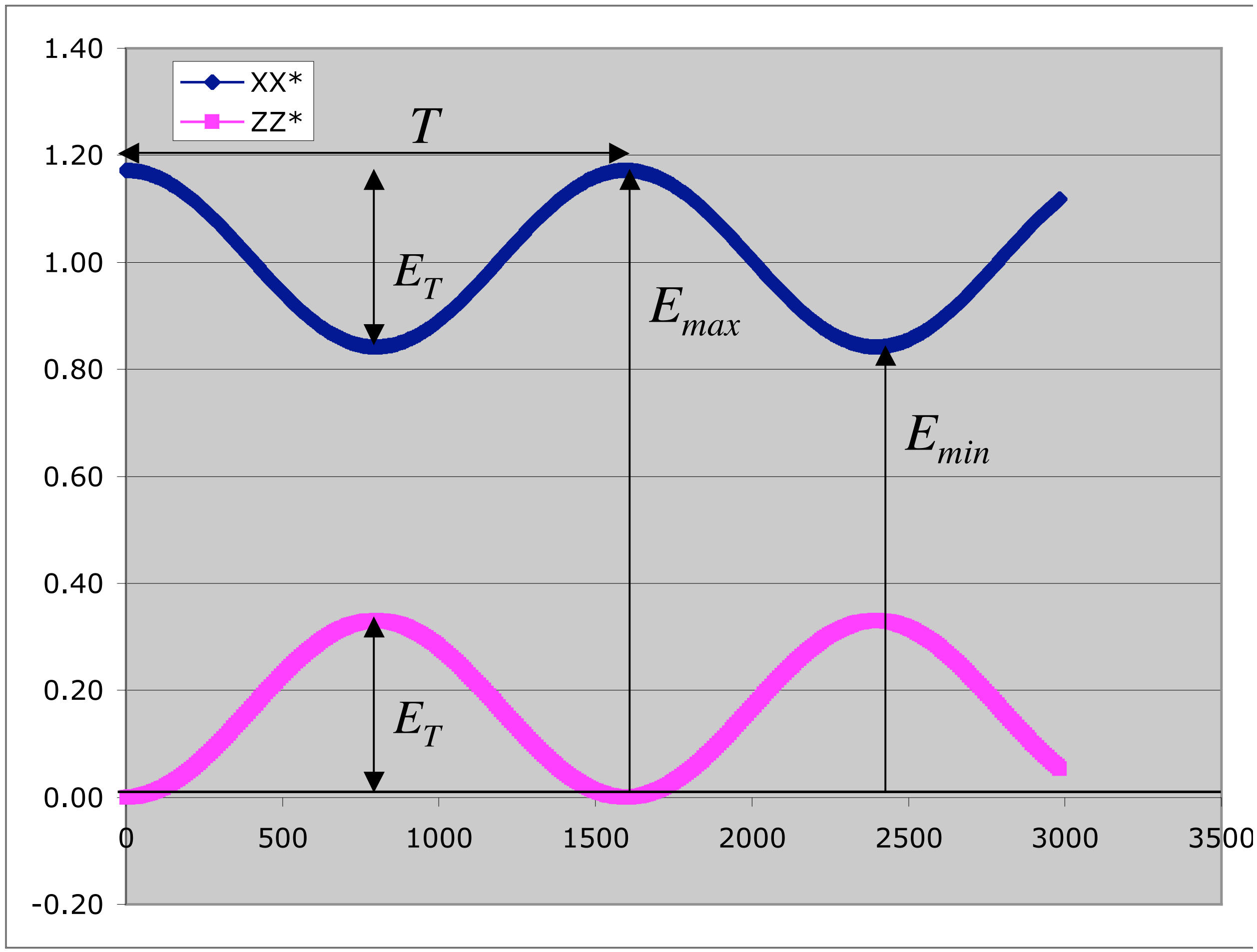
$Z=0$

and $E_{max} = |X|^2$

This gives:

$$\boxed{B = -A\frac{(\eta-\delta)\kappa}{k}}$$

$$\boxed{E_T = 4|AB^*|\frac{(\eta-\delta)\kappa}{k}}$$



Kick in the horizontal plane

This gives a direct method for measuring |C| by measuring the interchange period T

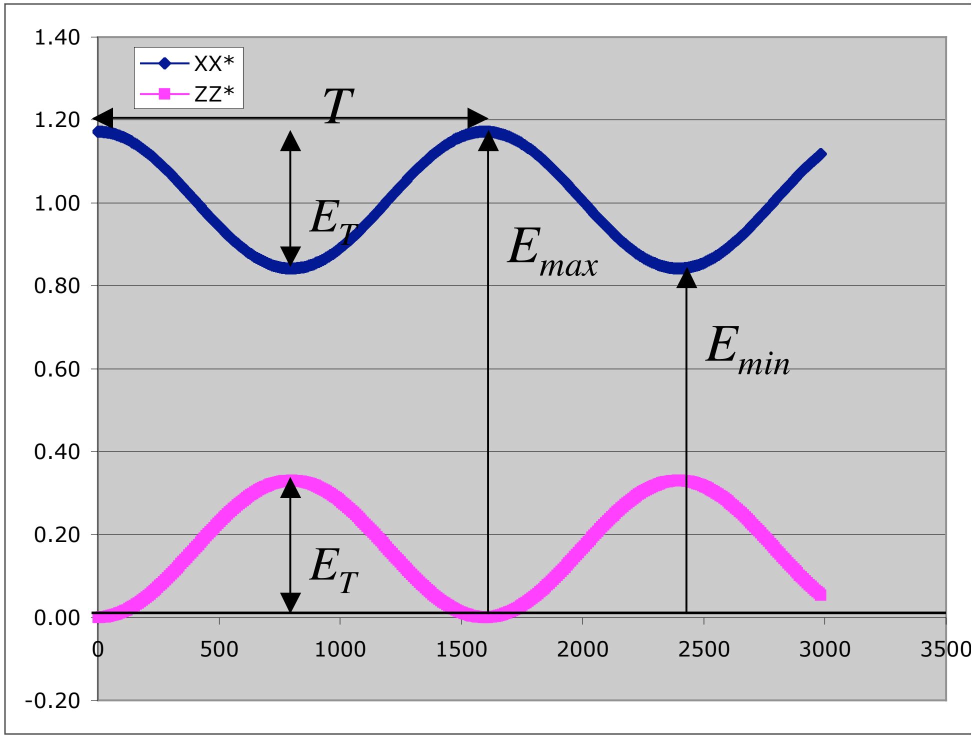
$$T = \frac{1}{f_{rev}} \frac{1}{\sqrt{\Delta^2 + |C|^2}}$$

And the modulation S

$$S = \frac{E_{\min}}{E_{\max}} = \frac{\Delta^2}{\Delta^2 + |C|^2}$$

The separation Δ of the uncoupled tunes cannot be measured but can be obtained from the above equations:

$$|C| = \frac{\sqrt{1-S}}{f_{rev}T} \quad ; \quad |\Delta| = \frac{\sqrt{S}}{f_{rev}T}$$



Minimum tune distance

Using $\kappa_x = \kappa + \delta/2$, $\kappa_z = \kappa - \delta/2$ the equations for x and z can be written as:

$$\boxed{\begin{aligned} x &= Ae^{i(\kappa+\eta/2)s} - B\frac{(\eta-\delta)\kappa}{k}e^{i(\kappa-\eta/2)s} \\ z &= Be^{i(\kappa-\eta/2)s} + A\frac{(\eta-\delta)\kappa}{k}e^{i(\kappa+\eta/2)s} \end{aligned}}$$

The motion is a combination of two normal modes with different oscillation frequencies:

$$Q_1 = R(\kappa+\eta/2) \; ; \; Q_2 = R(\kappa+\eta/2)$$

$$\boxed{\begin{aligned} Q_1 &= Q + \frac{1}{2}\sqrt{\Delta^2 + C^2} \\ Q_2 &= Q - \frac{1}{2}\sqrt{\Delta^2 + C^2} \end{aligned}}$$

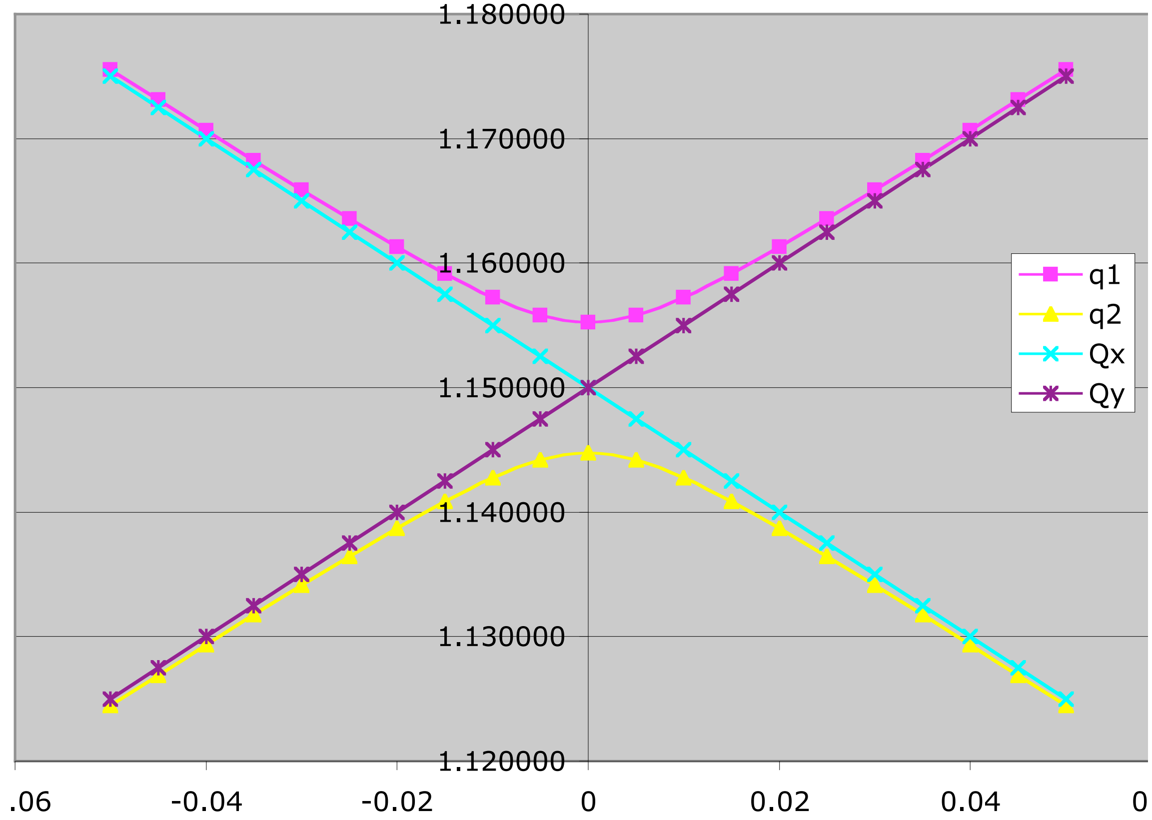
Minimum tune distance

Measured tunes as a function of the distance of the uncoupled tunes

When Δ is large the measured tunes are similar to the uncoupled Q_x and Q_z

Approaching the resonance (Δ=0) the two tunes will show a minimum approaching distance which is a measure of the coupling strength C.

This corresponds to full coupling, when the 2 tunes become undistinguishable



Coupling Coefficient

An exact analysis with the use of the Hamiltonian formalism and conjugate variables gives a rigorous result for the coupling coefficient in an alternating gradient lattice:

$$\boxed{C = \frac{1}{2\pi R}\int \sqrt{\beta_x \beta_z}\, K(\theta) \exp\left(i\left[(\mu_x - Q_x\theta) - (\mu_z - Q_z\theta) + \Delta\cdot\theta\right]\right) d\theta}$$

with

$$\boxed{K(\theta) = \frac{1}{2}\frac{R^2}{|B\rho|}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right) \;\; ; \;\; \theta = \frac{\mathrm{s}}{\mathrm{R}}}$$

Synchrotron motion

Synchrotron Radiation

- Emission of Synchrotron Radiation (SR) exerts a strong influence on electron beam dynamics in storage rings
- Emission of SR leads to damping of synchrotron and betatron oscillations and determines the beam sizes
- At present energies, these effects strongly affect the design of electron machines, while are negligible for proton machines
- For the next proton collider LHC, due to its very high energy, SR effects have to be taken into account
- In the following treatment we will refer to electrons

Radiated Power

SR is the energy emitted by a relativistic particle in motion on a circular trajectory

$$\beta = \frac{v}{c} \approx 1 \qquad ; \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} >> 1$$

The instantaneous rate of power emitted by SR is:

$$\boxed{P = \frac{2}{3}\frac{e^2 c}{4\pi\varepsilon_0}\frac{\beta^4\gamma^4}{\rho^2} = \frac{e^2 c^3}{2\pi} C_\gamma E^2 B^2} \qquad C_\gamma = 8.85 \cdot 10^{-5} GeV^{-3} m$$

SR is emitted on a broad frequency spectrum and in a narrow cone of aperture ~1/γ with respect to the electron velocity

Energy Loss Per Turn

$$\boxed{U_0 = \oint \frac{P}{\beta c} ds = \frac{C_\gamma}{2\pi} E^4 \oint \frac{ds}{\rho^2}}$$

$$\boxed{U_0 = \frac{C_\gamma}{2\pi} E^4 I_2}$$

$$\boxed{I_2 = \oint \frac{1}{\rho^2} ds}$$

I_2 is the second radiation integral

It is useful to relate the radius of curvature ρ, the bending magnet field B and the energy:

$$E\,[\mathrm{GeV}] = .3\, B\rho\,[\mathrm{Tm}]$$

For an iso-magnetic lattice (uniform bending radius):

$$\boxed{U_0[eV] = 8.85 \cdot 10^4 \frac{E^4[GeV]}{\rho[m]}}$$

Energy loss per turn and related parameters for various electron storage rings

	E (GeV)	ρ (m)	L (m)	T_0 (μs)	$U_{0,dip}$* (MeV)
Adone	.51	5	105	.35	.001
DAΦNE	.51	1.4	98	.31	.004
PEP B LE	3.1	30.5	2200	13.6	.27
PEP B HE	9.0	165	2200	13.6	3.5
LEP	100.	3100	$3\ 10^4$	89	2855

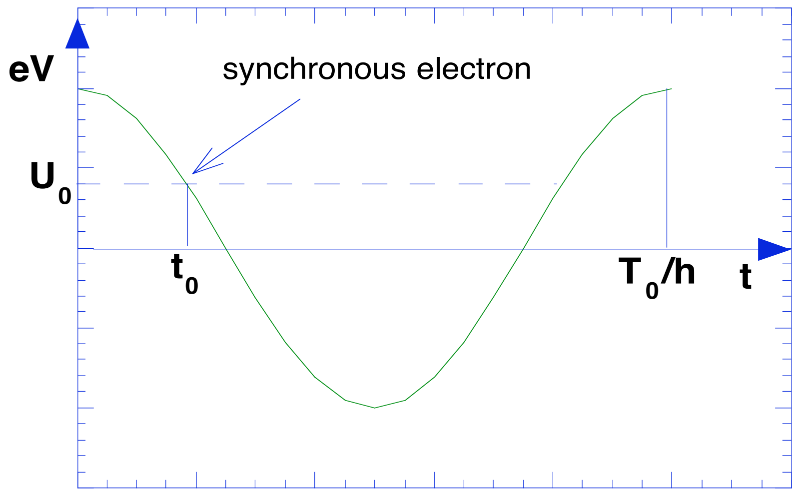
The same quantities for the next proton storage ring

	E (GeV)	ρ (m)	L (m)	T_0 (s)	$U_{0,dip}$ (MeV)
LHC	7700	2568	$3\ 10^4$	89	.011

* dip = from dipoles, excluding contributions from wigglers

Synchrotron Oscillations

The energy lost by SR has to be replaced by means of the electric field in the Radio Frequency (RF) cavities.



The synchronous particle travels on the reference trajectory (closed orbit of length L) with energy E_0 and revolution period $T_0 = 1/f_0 = L/c$

$$\boxed{E_0[GeV] = \frac{.3}{2\pi}\oint_L Bds[Tm]}$$

The RF frequency must be a multiple of the revolution frequency:

$$f_{RF} = hf_0$$

The synchronous particle arrives at the RF cavity at time t_0 so that the energy gained is equal to the energy U_0 lost per turn by SR

Synchrotron Oscillations

The arrival time for an off-momentum particle is given by the momentum compaction α_c

$$\boxed{\frac{\Delta T}{T_0} = \frac{\Delta L}{L} = \alpha \frac{\varepsilon}{E_0}}$$

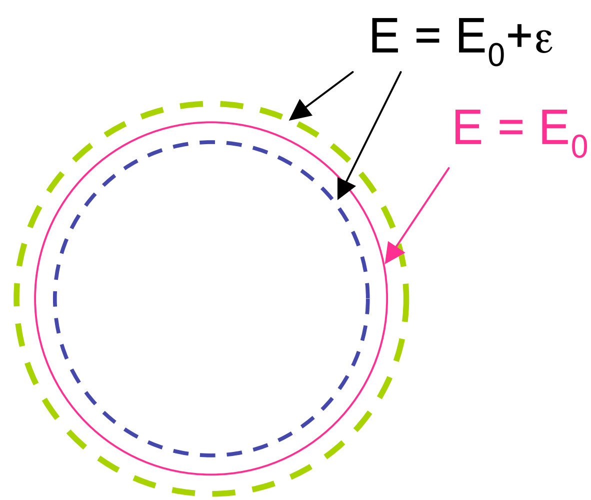
$$\boxed{\alpha_c = I_1/L}$$

The momentum compaction is generally positive

$\varepsilon = 0, t = t_0 \Rightarrow \Delta e = eV(t_0) = U_0$

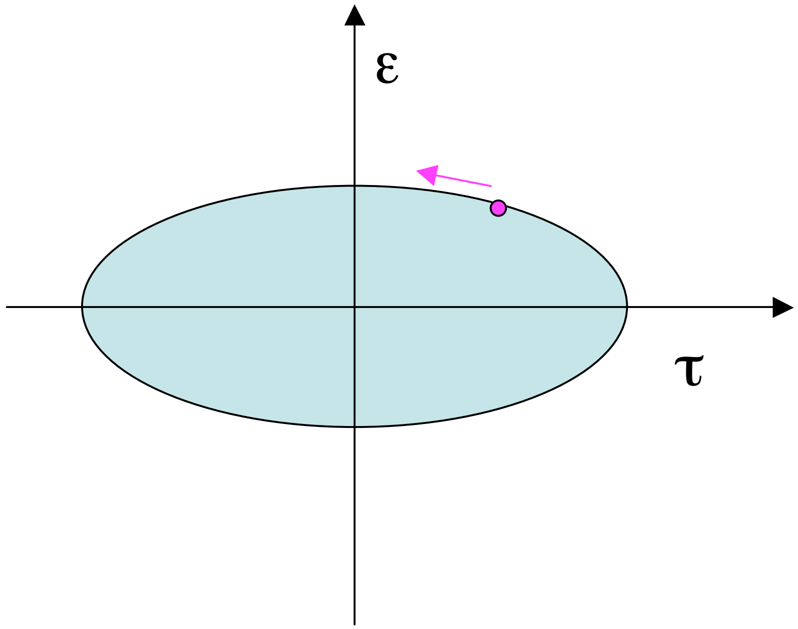
$\varepsilon > 0, t > t_0 \Rightarrow \Delta e = eV(t) < U_0$

$\varepsilon < 0, t < t_0 \Rightarrow \Delta e = eV(t) > U_0$



Synchrotron Oscillations

The particles oscillate in energy and phase with respect to the synchronous particle, which is at the bunch center



Let $\varepsilon = \Delta E/E$ and $\tau = t - t_0$

In the longitudinal phase space **ε, τ** the motion is represented by a point moving on an ellipse

The period of the synchrotron oscillation is typically many turns, much longer than that of the betatron oscillation (a fraction of a turn).

Synchrotron Oscillations

$\tau = \Delta s/c > 0$ is the time distance for an e^- ahead of the synchronous particle

Assuming that changes in ε and τ occur slowly with respect to T_0:

$$\boxed{\frac{d\tau}{dt} = -\alpha \frac{\varepsilon}{E_0}}$$

On average in one turn:

$$\boxed{\frac{d\varepsilon}{dt} = \frac{eV(\tau) - U(\varepsilon)}{T_0}}$$

For small oscillations we assume a linear RF voltage:

$$\boxed{eV(\tau) = U_0 + e\dot{V}_0\tau}$$

Synchrotron Oscillations

Combining these equations we obtain the usual equation of harmonic motion for the energy oscillations with an additional damping term

$$\frac{d^2\varepsilon}{dt^2} + 2\alpha_\varepsilon \frac{d\varepsilon}{dt} + \Omega^2\varepsilon = 0$$

$$\alpha_\varepsilon = \frac{1}{2T_0}\frac{dU}{d\varepsilon}$$

$$\Omega^2 = \frac{e}{T_0}\dot{V}_0 \frac{\alpha}{E_0}$$

$$\dot{V}_0 = \omega_{RF} V_0 \cos\phi_s$$

$$\sin\phi_s = \frac{U_0}{V_0}$$

The solution is:

$$\varepsilon(t) = A e^{-\alpha_e t}\cos(\Omega t - \phi)$$

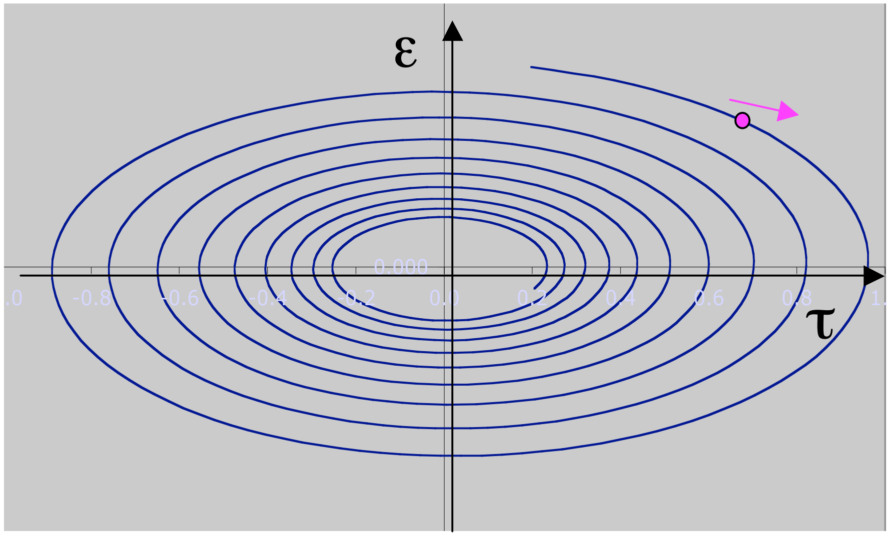
$$\tau(t) = \frac{-\alpha}{E_0 \Omega} A e^{-\alpha_e t}\sin(\Omega t - \phi)$$

Damping of Synchrotron Oscillations

Without damping the particle moves along the ellipse

The energy gained by the cavity is on average equal to U_0.

As the derivative $dU/d\varepsilon$ is proportional to ε the energy loss is larger than U_0 for $\varepsilon>0$ and smaller for $\varepsilon<0$.



Therefore the area of the ellipse decreases slowly and the trajectory spirals toward the inside

The oscillation amplitude decreases proportionally to $dU/d\varepsilon$

Damping of Synchrotron Oscillations

The rate of energy loss changes with energy because

- it is itself a function of energy
- the orbit deviates from the reference orbit and there may be a change in path length

$$\boxed{U(\varepsilon) = \frac{1}{c}\oint P\,dl}$$

P is a function of E^2 and B^2

$$\boxed{P = P_0 + \frac{2P_0}{E_0}\varepsilon}$$

and

$$\boxed{\frac{dU(\varepsilon)}{d\varepsilon} = \frac{1}{c}\oint \frac{2P_0}{E_0}\,ds = \frac{2U_0}{E_0}}$$

Damping Time

Taking into account also the path lengthening the damping coefficient is:

$$\boxed{\alpha_\varepsilon = \frac{1}{2T_0}\frac{dU}{d\varepsilon} = \frac{1}{2T_0}\frac{U_0}{E_0}\left(2 + \mathcal{D}\right)}$$

$$\mathcal{D} = \frac{\oint D(1-2n)/\rho^3 ds}{\oint 1/\rho^2 ds} = I_4/I_2$$

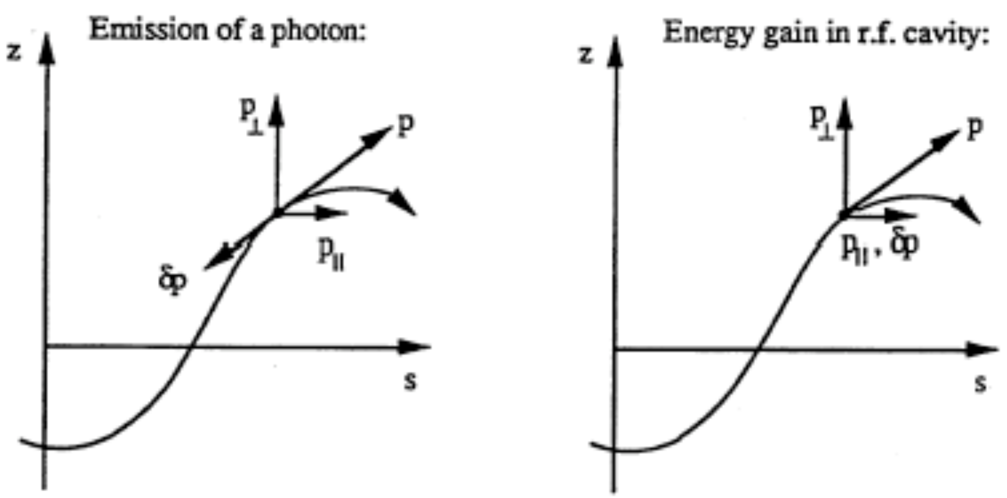
For separated function lattice $\mathcal{D} << 1$:

$$\boxed{\tau_\varepsilon = \frac{1}{\alpha_\varepsilon} \approx \frac{T_0 E_0}{U_0}}$$

τ_ε is the time in which the particle radiates all its energy

Damping of Vertical Betatron Oscillations

Effect of energy loss due to photon emission and energy gain in the RF cavity



The vertical oscillation is:

$$z = A\sqrt{\beta}\cos(\phi(s) + \phi_0)$$

$$z' = \frac{A}{\sqrt{\beta}}\sin(\phi(s) + \phi_0)$$

With amplitude A

$$A^2 = \gamma z^2 + 2\alpha z z' + \beta z'^2$$

After the RF cavity, since **$z' = p_\perp/p$**, it undergoes a variation : $\boxed{\delta z' = -z'\frac{\delta\varepsilon}{E_0}}$

The amplitude A varies by: $\frac{\langle \delta A\rangle}{A} = -\frac{1}{2}\frac{\delta\varepsilon}{E_0}$ averaged over one turn $\frac{\Delta A}{A} = -\frac{U_0}{2E_0}$

The damping decrement is $\boxed{\alpha_z = \frac{U_0}{2E_0T_0}}$

Damping partition

In the horizontal plane the damping coefficient has an additional term which accounts for the path length variation. In general:

$$\boxed{\alpha_i = \frac{J_i U_0}{2E_0 T_0}}$$

$i = x,z$ or ε and J_i are the damping partition numbers:

$J_x = 1 - D \quad ; \quad J_z = 1 \quad ; \quad J_\varepsilon = 2 + D \quad ; \quad D = I_4/I_2$

The sum of the damping rates for the three planes is a costant:

$$J_x + J_z + J_\varepsilon = 4$$

For damping in all planes simultaneously:

all $J_i > 0$ and hence $-2 < D < 1$

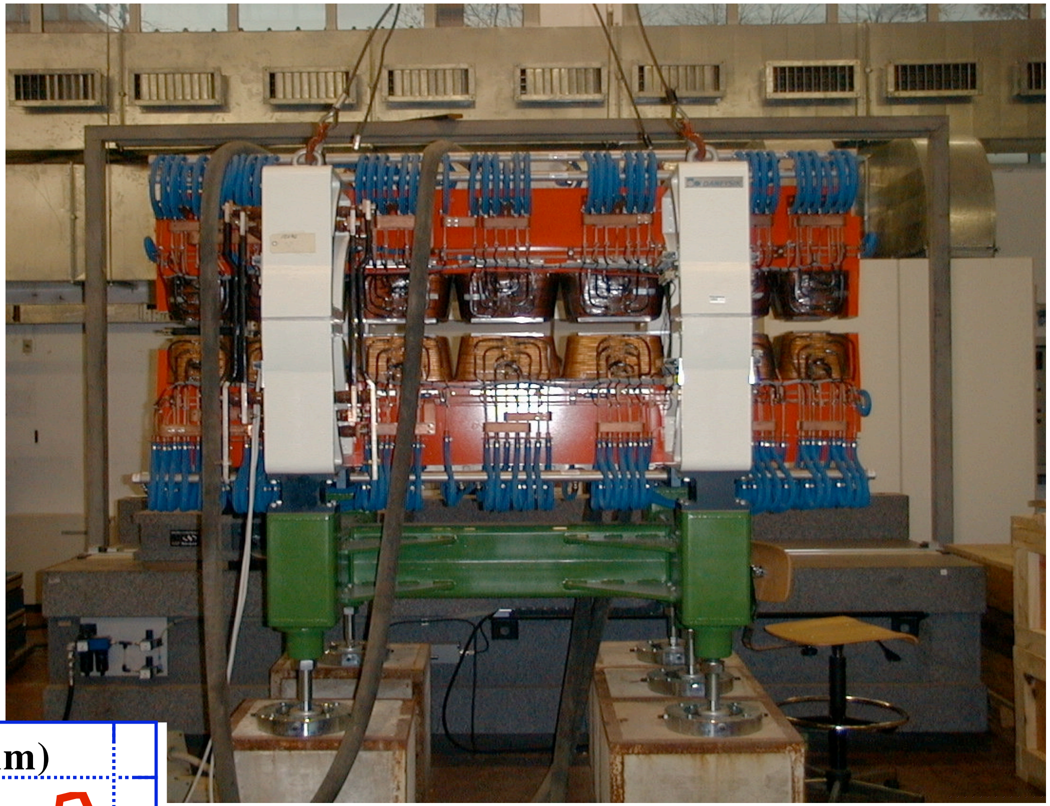
Radiation Damping Effects

- Equilibrium beam sizes
- Multi-cycle injection
- Damping rings
- Counteracts the beam instabilities
- Influence of S.R. Emission on Machine Design
  - RF system
  - vacuum system: heating and gas desorption
  - radiation damage
  - radiation background in collider experiments
- High Energy Storage Rings
  - The radius of curvature is increased to keep the power of the emitted radiation below an acceptable level $\rho \propto E^2$
- Low Energy Storage Rings
  - It is often useful to insert in the ring special devices, wiggler magnets, in order to increase the power emitted by SR and reduce the damping times

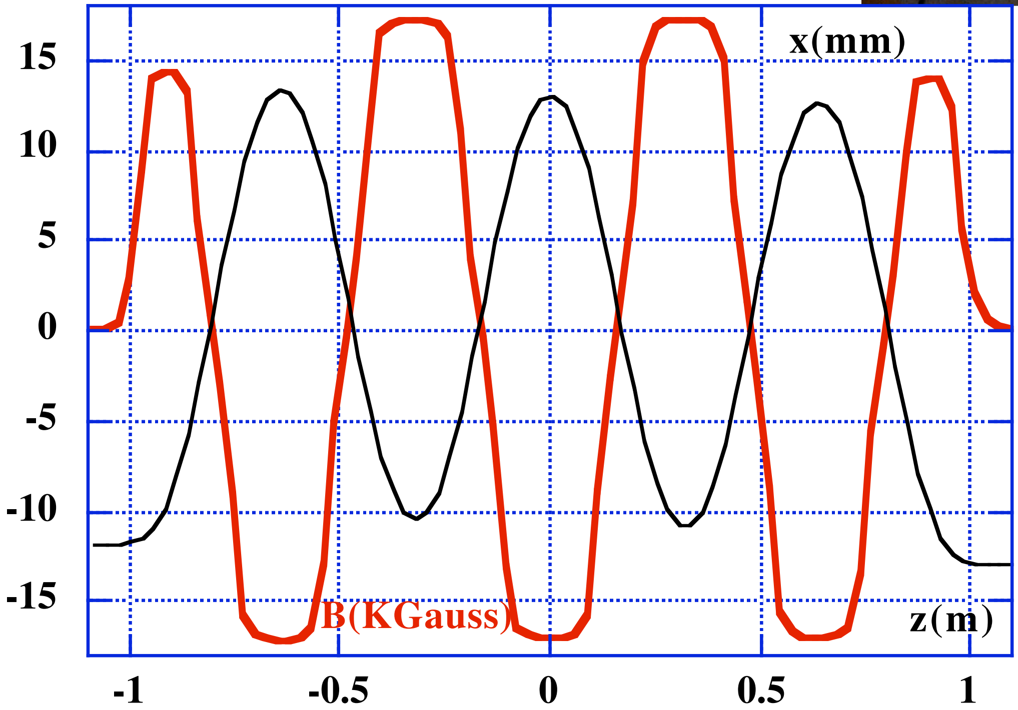
Wiggler Magnets

- A wiggler magnet is made of a series of dipole magnets with alternating polarity so that the total bending angle (i.e. the field integral along the trajectory) is zero.
- This device can be inserted in a straight section of the ring with minor adjustments of the optical functions.
- The damping time becomes faster because U_0 increases.
- Any number of periods can be added in order to get the desired damping time.

DAΦNE wiggler



Field and Trajectory



	dipoles	wigglers
B (T)	1.2	1.8
ρ (m)	1.4	.94
L (m)	8.8	8
U_0 (KeV)	4.3	5.0

Energy Oscillation Parameters for Various Electron Storage Rings

$$\boxed{U_0 = U_{0,\,dip} + U_{0,\,wig} = C_\gamma E^4 \left(\frac{1}{\rho_d} + \frac{1}{2\pi} \int_0^{L_w} \frac{ds}{\rho_w^2} \right)}$$

$$\tau_\varepsilon = \frac{T_0 E_0}{U_0}$$

	E (GeV)	$U_{0,dip}$ (MeV)	U_0 (MeV)	T_0 (μs)	T_{synch} (ms)	τ_ε (ms)	τ_ε / T_0
Adone	.51	.001	.001	.35	.05	180	$5\ 10^5$
DAΦNE	.51	.004	.009	.31	.03	18	$6\ 10^4$
PEP B LE	3.1	.27	1.24	7.3	.15	18	$2.5\ 10^3$
PEP B HE	9.0	3.5	3.5	7.3	.14	19	$2.6\ 10^3$
LEP	100.	2855	2855	89	1	3.1	35

Quantum Excitation and Beam Dimensions

Radiation damping is related to the *continuous* loss and replacement of energy.

Since the radiation is *quantized,* the statistical fluctuations in the energy radiated per turn cause a growth of the oscillation amplitudes.

The equilibrium distribution of the particles results from the combined effect of quantum excitation and radiation damping.

Mean Square Energy Deviation

The invariant oscillation amplitude is

$$\boxed{A^2 = \varepsilon^2(t) + \frac{E_0\Omega}{\alpha}\tau^2(t)}$$

When a photon of energy u is emitted the change in A^2 is:

$$\boxed{\delta A^2 = -2\varepsilon u + u^2}$$

and the total rate of change of A^2:

$$\frac{dA^2}{dt} = -\frac{2A^2}{\tau_\varepsilon} + \left\langle \mathcal{N}\left\langle u^2\right\rangle\right\rangle$$

The equilibrium is reached for $dA^2/dt = 0$ and the mean-square energy deviation is:

$$\left\langle \varepsilon^2\right\rangle = \frac{\left\langle A^2\right\rangle}{2} = \frac{\tau_\varepsilon}{4}\left\langle \mathcal{N}\left\langle u^2\right\rangle\right\rangle$$

Radiation Emission

The radiation is emitted in photons with energy $u = \hbar\omega$

The total number of photons emitted per electron per second is:

$$N = \int n(u)\,du = \frac{15\sqrt{3}}{8}\frac{P}{u_c} \quad ; \quad u_c = \hbar\omega_c \quad , \quad n(u) = \frac{P}{u_c^2}\frac{S(u/u_c)}{(u/u_c)}$$

The mean photon energy is:

$$\boxed{\langle u \rangle = \frac{\int u \cdot n(u)\,du}{N} = \frac{P}{N} = \frac{8}{15\sqrt{3}}u_c}$$

And the mean square photon energy is:

$$\boxed{\langle u^2 \rangle = \frac{\int u^2 \cdot n(u)\,du}{N} = \frac{11}{27}u_c^2}$$

Beam Energy Spread

- The energy deviation at a given time can be considered as the sum of all the previous photon emissions, and all the energy gains in the RF cavities.
- This sum contains a large number of statistically independent small terms.
- Therefore, for the Central Limit Theorem, the distribution of the energy deviation is Gaussian with standard deviation σ_ϵ.

$$\boxed{\sigma_\varepsilon^2 = \langle \varepsilon^2 \rangle = \frac{55}{32\sqrt{3}} \hbar c \gamma^3 \frac{\langle 1/\rho^3 \rangle}{\langle 1/\rho^2 \rangle} \frac{E_0}{J_\varepsilon}}$$

And the relative energy deviation

$$\left(\frac{\sigma_\varepsilon}{E_0} \right)^2 = C_q \frac{\gamma^2}{J_\varepsilon} \frac{\langle 1/\rho^3 \rangle}{\langle 1/\rho^2 \rangle} = C_q \gamma^2 \frac{I_3}{J_\varepsilon I_2}$$

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.84 \cdot 10^{-13} m$$

Bunch Length

A Gaussian distribution in energy results in a similar distribution in τ with standard deviation:

$$\boxed{\sigma_\tau = \frac{\alpha}{\Omega E_0}\sigma_\varepsilon}$$

with

$$\Omega^2 = \frac{2\pi\alpha}{T_0^2 E_0} heV_0 \cos\phi_s$$

$$\sin\phi_s = \frac{U_0}{V_0}$$

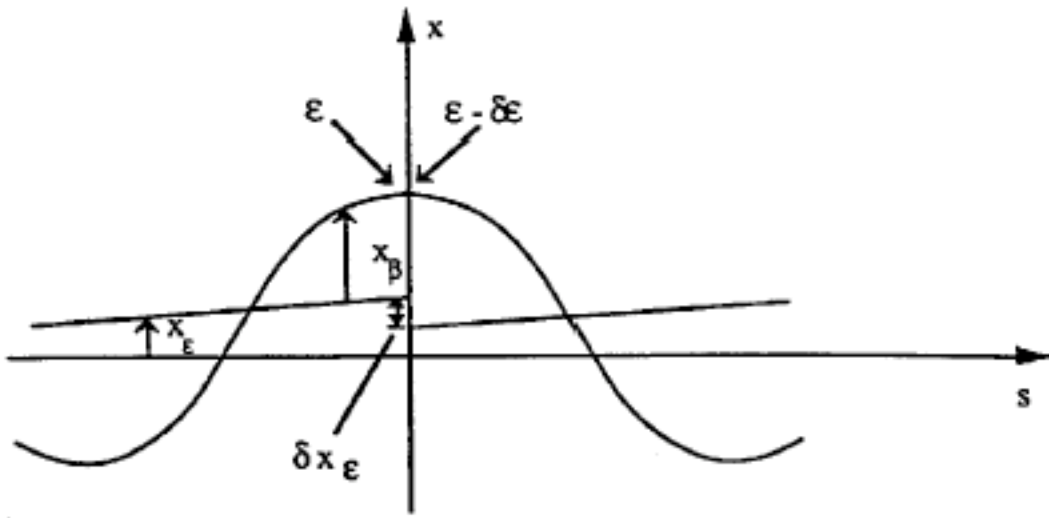
α_c momentum compaction, depends on lattice

Ω synchrotron frequency

V_0 RF peak voltage

Beam Emittance

Horizontal plane



Change of off-energy orbit and betatron amplitude in the H plane due to energy loss

$$\delta x_\beta = -D(s)\frac{u}{E_0}$$

$$\delta x'_\beta = -D'(s)\frac{u}{E_0}$$

The betatron oscillation invariant is:

$$\boxed{A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2}$$

and the change due to photon emission is:

$$\boxed{\delta A^2 = \left(\gamma D^2 + 2\alpha D D' + \beta D'^2\right)\frac{u^2}{E_0^2} = H(s)\frac{u^2}{E_0^2}}$$

Horizontal Emittance

The average rate of increase of A^2 is:

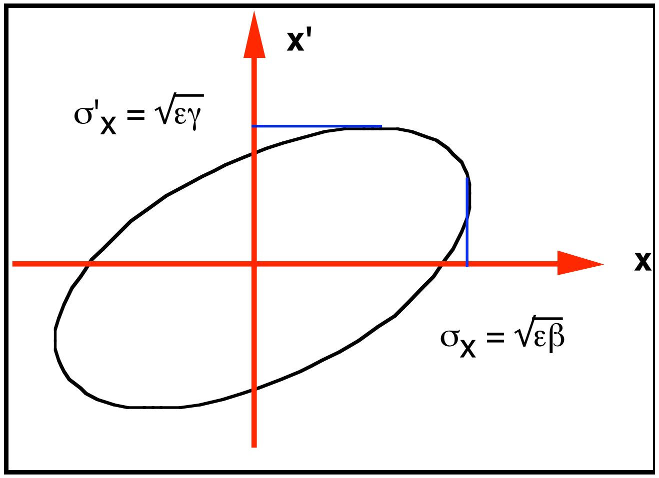
$$\boxed{\left\langle \frac{dA^2}{dt} \right\rangle = \frac{\left\langle \mathrm{N} \quad \langle u^2 \rangle \mathrm{H} \right\rangle}{E_0^2}}$$

Equating the excitation rate to the radiation damping the equilibrium mean square value is obtained. This defines the beam emittance ε_x:

$$\boxed{\varepsilon_x = \frac{\langle A^2 \rangle}{2} = C_q \frac{\gamma^2}{J_x} \frac{\langle \mathrm{H} / \rho^3 \rangle}{\langle 1/\rho^2 \rangle}} = C_q \frac{\gamma^2}{J_x} \frac{I_5}{I_2}$$

Emittance and beam sizes

$$\boxed{\varepsilon_x = C_q \frac{\gamma^2}{J_x}\frac{I_5}{I_2}}$$



$$I_5 = \oint \frac{H}{|\rho|^3} ds \quad ; \quad H = \gamma D_x^2 + 2\alpha D_x D'_x + \gamma D'^2_x$$

The emittance is constant for a given lattice and energy.
The projection of the distribution on the x and x' axis respectively is Gaussian with rms :

$$\boxed{\sigma_x = \sqrt{\varepsilon\beta(s)}} \qquad \boxed{\sigma'_x = \sqrt{\varepsilon\gamma(s)}}$$

At locations in the lattice where dispersion is non zero the contribution of the synchrotron motion is added in quadrature

$$\boxed{\sigma_{x\varepsilon} = D\frac{\sigma_\varepsilon}{E_0}} \qquad \boxed{\sigma'_{x\varepsilon} = D'\frac{\sigma_\varepsilon}{E_0}}$$

Vertical Emittance

- Generally storage rings lie in the horizontal plane and have no bending and no dispersion in the vertical plane.
- A very small vertical emittance arises from the fact that the photons are emitted at a small angle with respect to the direction of motion ($\theta_{rms} \approx 1/\gamma$)
- The resulting vertical equilibrium emittance is:

$$\boxed{\varepsilon_z = \frac{C_q}{2J_z} \frac{\langle \beta_z / \rho^3 \rangle}{\langle 1/\rho^2 \rangle}}$$

- This vertical emittance can be generally neglected. For the ILC DR $\langle \beta_z \rangle / \rho \sim 40/100$ and $\varepsilon_z \sim 8\ 10^{-14}$, which is 4% of the design vertical emittance, not completely negligible

Vertical Emittance

In practice a contribution to the vertical emittance can come from:

- coupling of horizontal and vertical betatron oscillations due to:
  - skew quadrupole field errors (angular errors in the quadrupole alignment and vertical orbit in the sextupoles)
  - errors in the compensation of detector solenoids

- vertical dispersion due to:
  - angular errors in the dipole alignment
  - vertical orbit in the quadrupoles

Vertical Emittance

If the vertical emittance depends on a large number of small errors randomly distributed along the ring, it can be described in terms of a coupling coefficient κ

$$\boxed{\varepsilon_x = \frac{1}{1+\kappa}\varepsilon_0 \quad ; \quad \varepsilon_z = \frac{\kappa}{1+\kappa}\varepsilon_0}$$

$0 < \kappa < 1$

The sum of the horizontal and vertical emittances is constant, often called "natural beam emittance"

Effect of Damping Wigglers on ε_x

- We have already seen that insertion of wigglers in a ring increases I_2 and therefore the energy radiated per turn
- The main effect is a reduction of the damping times

$$\boxed{U_0 = C_\gamma \frac{E^4}{\rho_a} + C_\gamma E^2 \frac{e^2 c^2}{2\pi} \int_0^{L_w} B_w^2 ds = U_a + U_w} \qquad \tau_y = 2\frac{E_0 T_0}{J_y U_0}$$

Wigglers can have a strong effect on emittance

We assume: $J_x \sim 1$ and $F_w = U_w/U_a$

$$\varepsilon_a = C_q \gamma^2 \frac{I_{5a}}{I_{2a}}$$ arc emittance

$$\varepsilon_w = C_q \gamma^2 \frac{I_{5w}}{I_{2w}}$$ wiggler emittance

$$\varepsilon_{x0} = \frac{\varepsilon_a}{1+F_w} + \varepsilon_w \frac{F_w}{1+F_w}$$

Effect of Damping Wigglers

$$\boxed{\varepsilon_{x0} = \frac{\varepsilon_a}{1+F_w} + \varepsilon_w \frac{F_w}{1+F_w}}$$

- If F_w>>1 the arc emittance is reduced by the factor F_w and the ring emittance is dominated by the wiggler
- Inserting the wigglers in a zero dispersion section the wiggler emittance can be made very small
- Therefore insertion of wigglers allows to reduce both the beam emittance and the damping time
- Wigglers are extensively used in damping rings: the ILC DR has 200 m of wigglers
- Wigglers in dispersive sections are sometimes used to increase the emittance in storage ring colliders

Summary of beam parameters related to the synchrotron radiation integrals

$$\alpha_c = \frac{I_1}{C}$$

$$U_0 = \frac{C_\gamma}{2\pi} E^4 I_2$$

$$\tau_i = \frac{2}{J_i}\frac{E_0 T_0}{U_0} \quad , \quad i = x, z, \varepsilon$$

$J_x = 1 - D$; $J_z = 1$; $J_\varepsilon = 2 + D$; $D = I_4/I_2$

$$\left(\frac{\sigma_\varepsilon}{E_0}\right)^2 = C_q \gamma^2 \frac{I_3}{J_\varepsilon I_2} \qquad C_q = 3.84 \cdot 10^{-13}\, m$$

$$\sigma_\tau = \frac{\alpha}{\Omega E_0}\sigma_\varepsilon$$

$$\Omega^2 = \frac{2\pi\alpha}{T_0^2 E_0} heV_0 \cos\phi_s \; ; \quad \sin\phi_s = \frac{U_0}{V_0}$$

$$\varepsilon_x = C_q \frac{\gamma^2}{J_x}\frac{I_5}{I_2} \qquad \varepsilon_z = \frac{C_q}{2J_z}\frac{\langle \beta_z / \rho^3 \rangle}{I_2}$$

SR only

$$I_1 = \oint \frac{D(s)}{\rho} ds$$

$$I_2 = \oint \frac{1}{\rho^2} ds$$

$$I_3 = \oint \frac{1}{\rho^3} ds$$

$$I_4 = \oint \frac{D(1-2n)}{\rho^3} ds$$

$$I_5 = \oint \frac{H}{|\rho|^3} ds$$

$$H = \gamma D_x^2 + 2\alpha D_x D'_x + \gamma D'^2_x$$

Lecture 5
Damping Ring Basics

Intrabeam Scattering

ILC Accelerator school

Intrabeam scattering

- Intrabeam scattering (IBS) is an elastic Coulomb scattering between pairs of particles within a bunch.
- The particles perform betatron oscillations in a plane perpendicular to the direction of motion
- Due to the scattering two particles can transform their transverse momenta into longitudinal momenta.
- When making the transformation from the reference frame moving with the bunch center of mass to the laboratory frame the longitudinal momentum is multiplied by the relativistic factor γ
- For highy relativistic particles (γ>>1) fraction of the longitudinal momentum gain or lost in the collision may be not negligible

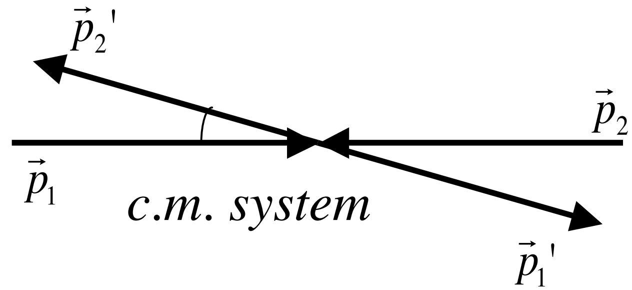
Touschek scattering

- If the new longitudinal momenta are outside the momentum acceptance of the ring they get lost
- This effect is called Touschek scattering since it was observed by Bruno Touschek at ADA the first prototype storage ring of LNF
- This effect can give a strong lifetime limitation
- For damping rings lifetime can be of the order of minutes
- This is not an operational limitation since the beam has to be stored in the ring for a very short time (200 ms for ILC DR)
- It could be a limitation in the commissioning phase when beams need to be stored to tune the ring parameters

Intrabeam Scattering

- If the momentum of the particle after the scattering is still inside the ring acceptance the result is an additional momentum spread in the bunch
- IBS can lead to growth in the transverse and longitudinal emittances. This could prevent achieving the very small beam sizes required in DRs

Calculation of Rise Times



- The theory of IBS has been derived by Piwinski and by Bjorken and Mitingwa
- The change of momenta of two particles due to the collision is calculated in the center of mass system of the two particles and then transformed back to the laboratory
- It is calculated the change of the amplitude of betatron and synchrotron oscillations
- Then it is taken the average over the scattering angles, weigthed with the cross section, and over momenta and position of the particles with Gaussian distribution

Rise Times Formulae

$$\frac{1}{\tau_p} = \frac{1}{2\sigma_p^2}\frac{d\sigma_p^2}{dt} = A\frac{\sigma_h^2}{\sigma_p^2} f(a,b,c)$$

$$\frac{1}{\tau_x} = \frac{1}{2\sigma_x^2}\frac{d\sigma_x^2}{dt} = A\left[f(\frac{1}{a},\frac{b}{a},\frac{c}{a}) + \frac{D^2\sigma_p^2}{\sigma_{x\beta}^2} f(a,b,c)\right]$$

$$\frac{1}{\tau_z} = \frac{1}{2\sigma_z^2}\frac{d\sigma_z^2}{dt} = Af(\frac{1}{a},\frac{b}{a},\frac{c}{a})$$

with

$$a = \frac{\sigma_h}{\gamma\sigma'_x},\quad b = \frac{\sigma_h}{\gamma\sigma'_z},\quad c = \beta\sigma_h\sqrt{\frac{d}{r_0}}$$

$$\frac{1}{\sigma_h^2} = \frac{1}{\sigma_p^2} + \frac{D^2}{\sigma_{x\beta}^2}$$

$$\boxed{A = \frac{r_0^2 c N_b}{64\pi\sigma_s\sigma_p\sigma_{x\beta}\sigma_{z\beta}\sigma'_x\sigma'_z\beta^3\gamma^4}}$$

$$f(a,b,c) = 8\pi\int_0^1\left[\ln(\frac{c^2}{2}(\frac{1}{\sqrt{p}}+\frac{1}{\sqrt{p}})) - 0.577\ldots\right](1-3x^2)\frac{dx}{\sqrt{pq}}$$

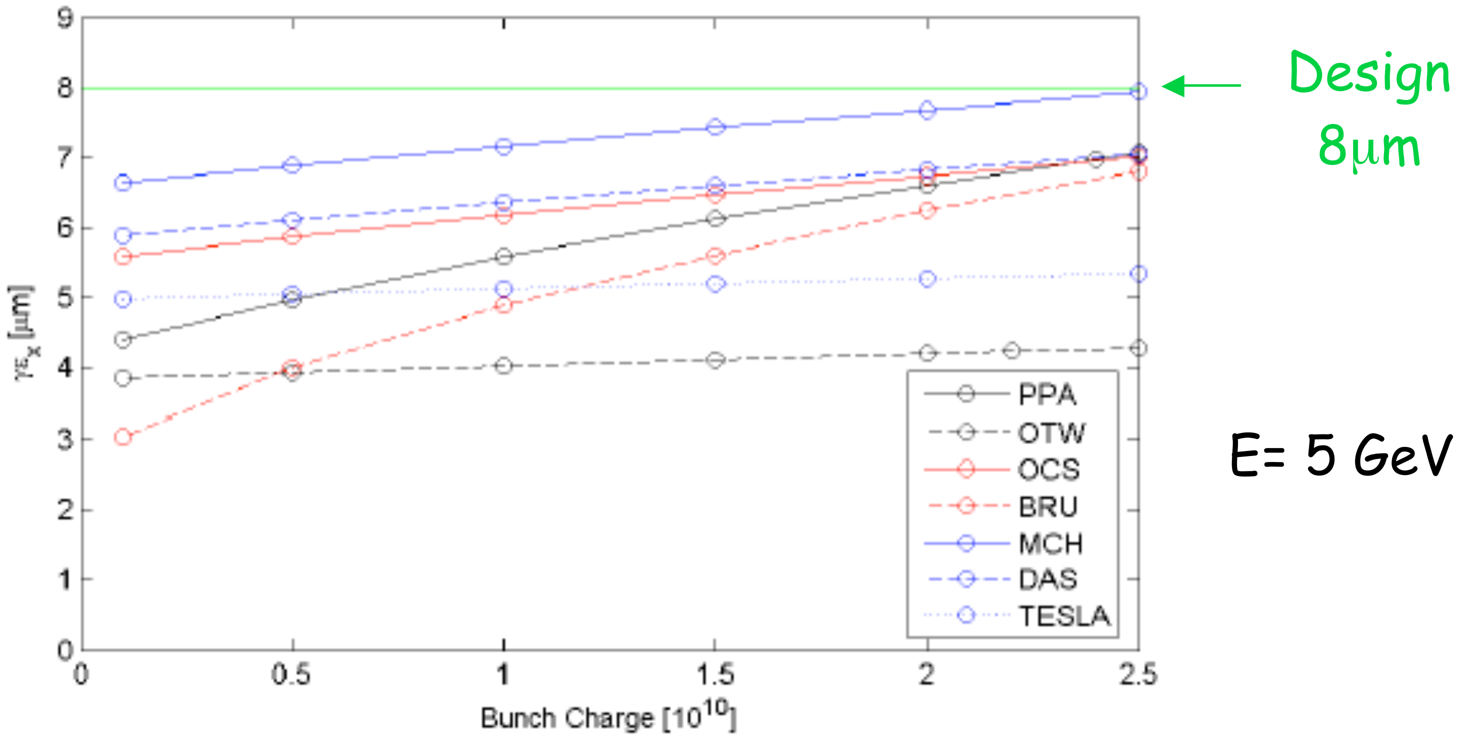
$$p = a^2 + x^2(1-a^2)\ ,\quad q = b^2 + x^2(1-b^2)$$

A. Piwinski, CERN 87/03

Intrabeam Scattering

- Beam emittances and energy spread result from the equilibrium between radiation damping and IBS rise times
- Rise times have a strong energy dependence($\tau_{p,x,z} \propto \gamma^4$)
- Rise times are proportional to the bunch density in phase space
- DR aim to ultra low emittances and high bunch charges and can be affected by IBS emittances growth
- Possible Cures:
- Lattice design - keep $D_x \neq 0$ in most of the ring and Dx=0 at the extraction point. So $\sigma_x = \sqrt{(\varepsilon\beta_x + D_x^2\sigma_p^2)}$ in most of the ring is larger than at the extraction. This reduces bunch density and can reduce IBS in low emittance lattices, where the invariant function H is small
- Increase radiation damping - in fact the emittance results from the equilibrium between radiation damping and IBS rise times.

Horizontal Emittance Growth From IBS in the ILC DR Reference Lattices

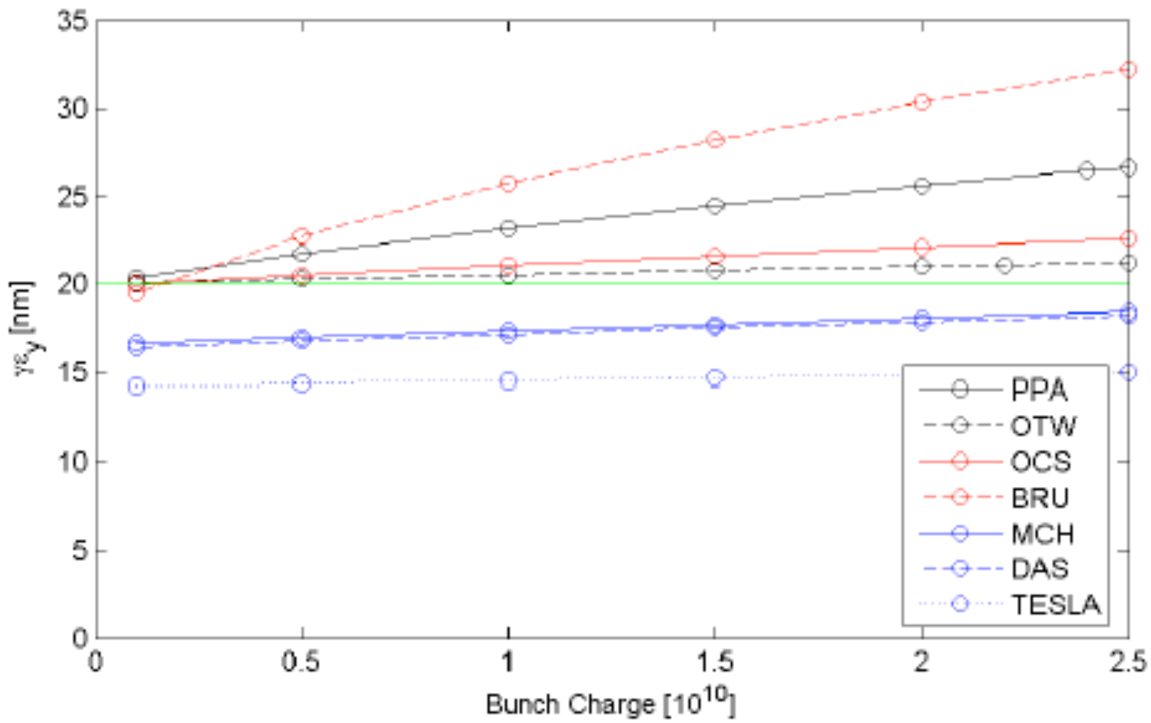


E= 5 GeV

The growth from IBS is a function of the optics, and is strongly dependent on the energy (BRU: E=3.74 GeV)

All the lattices meet the specification on the horizontal emittance

Vertical Emittance Growth From IBS in the ILC DR Reference Lattices



_ Design 20nm

E= 5 GeV

The growth is calculated assuming that vertical dispersion and betatron coupling contribute equally to the vertical emittance

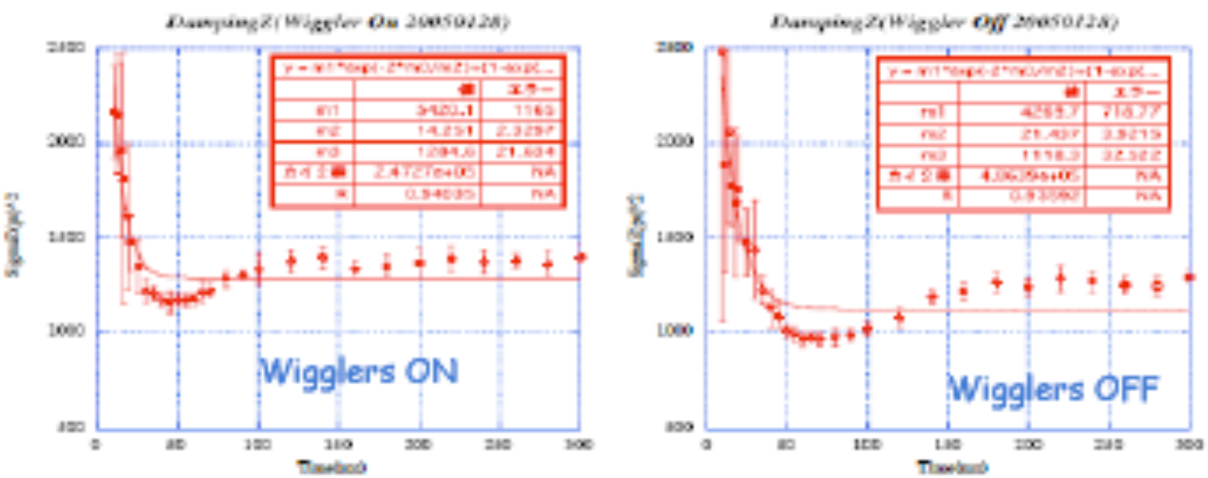
IBS at KEK-ATF

- The IBS effect is strongly dependent on the bunch density, and therefore becomes important for ultra low vertical emittance
- Experimental studies of IBS in electron storage rings have been performed in the KEK-ATF, which has achieved the lowest vertical emittance of any operating storage ring
- Calculations of the equilibrium emittances are in good agreement with the experimental results.

ATF with wigglers

ATF Damping times

Damping Time	Cal.,wiggler off	Cal.,wiggler on	Meas.wiggler off	Meas.wiggler on
Horizontal τ_x	17.5 ms	15.0 ms	19.3+/-0.63 ms	15.7+/-0.38 ms
Vertical τ_y	28.5 ms	23.0 ms	28.8+/-1.5 ms	25.4+/-0.67 ms
Longitudinal τ_z	20.5 ms	15.5 ms	21.4+/-3.9 ms	14.2+/-2.4 ms



- Four wigglers (2m long) were turned on
- Measured damping times and emittances found consistent with calculations

- Horizontal beam size, bunch length and energy spread growth, due to IBS, observed
- Reduction of damping time of emittance and of IBS effect is observed with wigglers on

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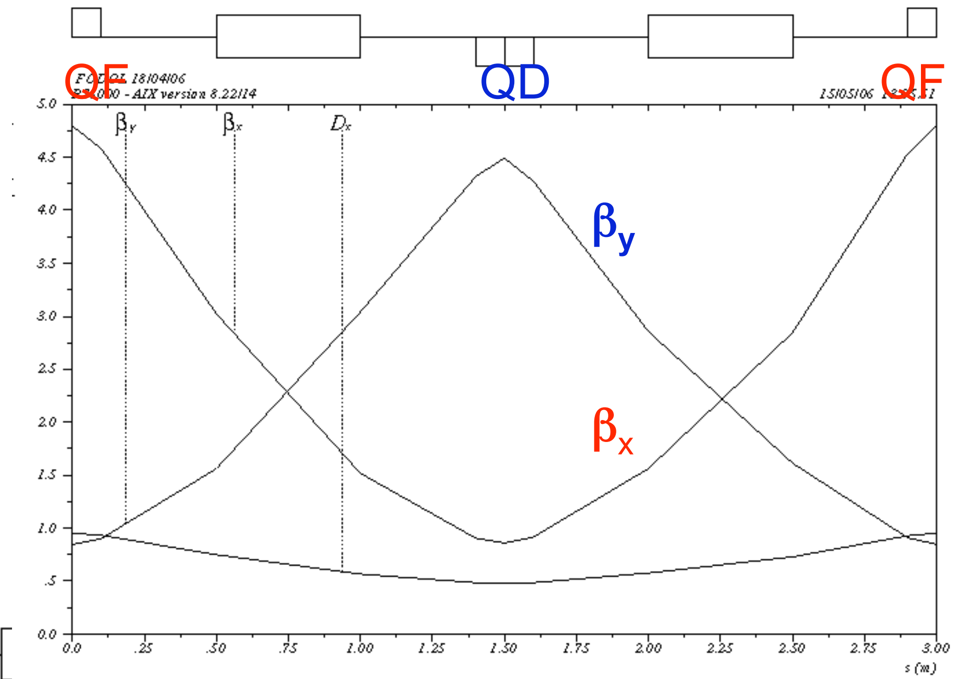
FODO Lattice

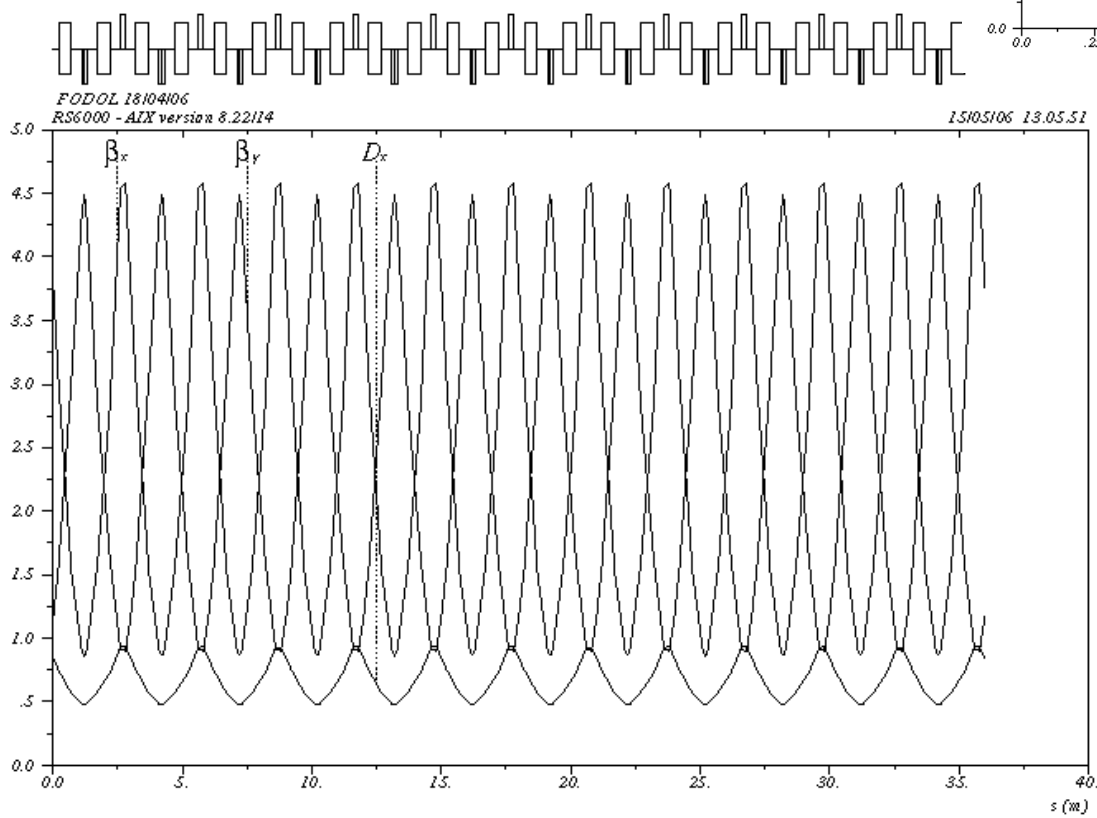
12 FODO cells

$Q_x = 3.15$, $Q_z = 3.15$



FODO cell

$\phi_x = \phi_y = 94.5^o$



Thin Lens FODO Cell

In thin lens approximation and representing dipoles as drift spaces the matrix of a FODO cell can be written as

$$M=\overset{\text{QF}}{\begin{pmatrix}1 & 0\\ -\frac{1}{2f} & 1\end{pmatrix}}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\overset{\text{QD}}{\begin{pmatrix}1 & 0\\ \frac{1}{f} & 1\end{pmatrix}}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\overset{\text{QF}}{\begin{pmatrix}1 & 0\\ -\frac{1}{2f} & 1\end{pmatrix}}=\begin{pmatrix}1-\dfrac{L^2}{2f^2} & 2L(1+\dfrac{L}{2f})\\ -\dfrac{L^2}{2f^2}(1-\dfrac{L}{2f}) & 1-\dfrac{L^2}{2f^2}\end{pmatrix}$$

Comparing this with the matrix of a periodic structure evaluate the betatron phase advance ϕ and the Twiss functions β_F and α_F at the center of QF (optional: at the center of QD, writing the matrix starting from QD center)

Optional: Observe the behaviour of β_F and β_D as a fuction of ϕ; what happens at 180º?